

Hanf number for Scott sentences of computable structures

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November 2, 2016

Abstract

The *Hanf number* for a set S of sentences in $\mathcal{L}_{\omega_1, \omega}$ (or some other logic) is the least infinite cardinal κ such that for all $\varphi \in S$, if φ has models in all infinite cardinalities less than κ , then it has models of all infinite cardinalities. S-D. Friedman asked what is the Hanf number for Scott sentences of computable structures. We show that the value is $\beth_{\omega_1^{CK}}$. The same argument proves that $\beth_{\omega^{CK}}$ is the Hanf number for Scott sentences of hyperarithmetical structures.

1 Introduction

Scott [11] showed that for any countable structure \mathcal{A} for a countable vocabulary, there is a sentence of $L_{\omega_1 \omega}$ whose countable models are exactly the isomorphic copies of \mathcal{A} . Such a sentence is called a *Scott sentence* for \mathcal{A} . In this paper, we show that the Hanf number for Scott sentences of computable structures is $\beth_{\omega_1^{CK}}$, where ω_1^{CK} is the first non-computable ordinal. We say that τ is a *computable vocabulary* if the set of symbols is computable, and there is a computable function giving the arities.

Definition 1. Let τ be a computable vocabulary, and let \mathcal{A} be a τ -structure with universe a subset of ω . The structure \mathcal{A} is *computable* if its atomic diagram, $D(\mathcal{A})$, is computable. We think of the elements as constants, and we identify sentences with their Gödel numbers, so that $D(\mathcal{A})$ is a subset of ω .

The paper splits into two parts. In Section 2, we prove the following theorem, which establishes $\beth_{\omega_1^{CK}}$ as an upper bound for the Hanf number for Scott sentences of computable structures.

Theorem 1.1. Let \mathcal{A} be a computable structure for a computable vocabulary τ , and let φ be a Scott sentence for \mathcal{A} . If φ has models of cardinality \beth_α for all $\alpha < \omega_1^{CK}$, then it has models of all infinite cardinalities.

For an infinite cardinal κ and an $L_{\omega_1 \omega}$ -sentence φ , we say that φ *characterizes* κ if φ has a model of cardinality κ , but not in cardinality κ^+ . In Section 3, we

exhibit specific examples of computable structures \mathcal{A}_a , corresponding to ordinal notations $a \in \mathcal{O}$, such that the Scott sentence of \mathcal{A}_a characterizes $\beth_{|a|}$, where $|a|$ is the ordinal with notation a . This is Theorem 3.1. Combining Theorems 1.1 and 3.1, we obtain the following.

Theorem 1.2. *The Hanf number for Scott sentences of computable structures is equal to $\beth_{\omega_1^{CK}}$.*

The Hanf number for Scott sentences of hyperarithmetical structures is also equal to $\beth_{\omega_1^{CK}}$. The proof that we give for Theorem 1.1 also shows that the Hanf number for Scott sentences of hyperarithmetical structures is at most $\beth_{\omega_1^{CK}}$, and the Scott sentences of computable structures witness that it is at least $\beth_{\omega_1^{CK}}$. (Similar reasoning would show that for a countable admissible set A with ordinal γ , the Hanf number for Scott sentences of structures in A is \beth_γ . We will not discuss this.)

In the remainder of the introduction, we give some conventions and basic definitions, and we recall some well-known results.

1.1 Background in infinitary logic

The following two results are given in [5]. The first result, proved independently by Morley and by López-Escobar, says that the Hanf number for $\mathcal{L}_{\omega_1, \omega}$ is \beth_{ω_1} .

Theorem 1.3 (Morley, López-Escobar). *Let Γ be a countable set of sentences of $\mathcal{L}_{\omega_1, \omega}$. If Γ has models of cardinality \beth_α for all $\alpha < \omega_1$, then it has models of all infinite cardinalities.*

The next result, proved independently by Morley and by Barwise, says that for a countable admissible set A with ordinal γ , the Hanf number for the admissible fragment $\mathcal{L}_A = A \cap \mathcal{L}_{\omega_1, \omega}$ is at most \beth_γ .

Theorem 1.4 (Morley, Barwise). *Let A be a countable admissible set with $o(A) = \gamma$, and let φ be a sentence of \mathcal{L}_A . If φ has models of cardinality \beth_α for all $\alpha < \gamma$, then it has models of all infinite cardinalities.*

The proofs of Theorems 1.3 and 1.4 use the Erdős-Rado Theorem to produce a model of φ with an infinite indiscernible sequence, in a language with added Skolem functions. The indiscernible sequence can be stretched to give models in arbitrarily large cardinalities.

We shall use “computable” infinitary formulas. The *computable infinitary formulas* are formulas of $\mathcal{L}_{\omega_1, \omega}$ in which the infinite disjunctions and conjunctions are over c.e. sets. To make this precise, we would assign indices to the formulas, based on notations in Kleene’s \mathcal{O} , as is done in [1]. The least admissible set that contains ω is $A = L_{\omega_1^{CK}}$. The subsets of ω in A are exactly the hyperarithmetical sets, and all computable (or hyperarithmetical) structures are elements of A . The computable infinitary formulas (in a fixed computable

vocabulary τ) are essentially the same as the $\mathcal{L}_{\omega_1, \omega}$ formulas in the admissible fragment \mathcal{L}_A ; that is, for any formula $\varphi(\bar{x})$ in \mathcal{L}_A , there is a computable infinitary formula $\psi(\bar{x})$ that is logically equivalent to $\varphi(\bar{x})$.

For many computable structures \mathcal{A} , there is a computable infinitary Scott sentence φ . By Theorem 1.4, if φ has models in all infinite cardinalities less than $\beth_{\omega_1^{CK}}$, then it has models of all infinite cardinalities. However, some computable structures do not have a computable infinitary Scott sentence. In particular, this is so for the “Harrison ordering”, a computable ordering of type $\omega_1^{CK}(1+\eta)$. The computable infinitary sentences true in the Harrison ordering are exactly those true in the ordering of type ω_1^{CK} . In fact, for any countable admissible set A , with ordinal α , there are structures in A with no Scott sentence in the admissible fragment \mathcal{L}_A . One such structure is an ordering of type $\alpha(1+\eta)$.

We do not use the notion of Scott rank in this paper, so we shall not give a definition. We mention, for general interest, a result of Nadel [6], [7], saying that for a computable, or hyperarithmetical, structure \mathcal{A} , there is a computable infinitary Scott sentence just in case the Scott rank is less than ω_1^{CK} . More generally, if the structure \mathcal{A} is an element of a countable admissible set A with ordinal γ , then it has a Scott sentence in \mathcal{L}_A just in case the Scott rank is less than γ . The result below follows from a general theorem of Ressayre [9],[10].

Theorem 1.5.

1. *If \mathcal{A} and \mathcal{B} are computable (or hyperarithmetical) structures satisfying the same computable infinitary sentences, then $\mathcal{A} \cong \mathcal{B}$.*
2. *If \mathcal{A} is a computable (or hyperarithmetical) structure, and \bar{a} and \bar{b} are tuples satisfying the same computable infinitary formulas in \mathcal{A} , then there is an automorphism of \mathcal{A} taking \bar{a} to \bar{b} .*

1.2 Fraïssé limits

The computable structures that we produce in Section 3 will be “Fraïssé limits.” In the discussion below, we will give slightly non-standard definitions. We will state a simple result on existence of computable Fraïssé limits that is not the most general, but is exactly suited to our needs.

Definition 2. *Let τ be a countable relational vocabulary. Let \mathbf{K} be a set of τ -structures, all finite.*

1. *\mathbf{K} satisfies the hereditary property, or HP, if for all $A \in \mathbf{K}$, all proper substructures of A are in \mathbf{K} .*
2. *\mathbf{K} satisfies the joint embedding property, or JEP, if for all $A, B \in \mathbf{K}$, there exists $C \in \mathbf{K}$ with embeddings $f : A \rightarrow C$ and $g : B \rightarrow C$.*
3. *\mathbf{K} satisfies the amalgamation property, or AP, if for all $A, B, C \in \mathbf{K}$ with embeddings $f : C \rightarrow A$ and $g : C \rightarrow B$, there is some $D \in \mathbf{K}$, with embeddings $f' : A \rightarrow D$ and $g' : B \rightarrow D$, such that $f' \circ f = g' \circ g$.*

4. \mathbf{K} is an age if it satisfies *HP*, *JEP*, and *AP*.

Remarks.

1. For Fraïssé, the vocabulary of an age may have function symbols, and the structures making up the age are finitely generated, but not necessarily finite. For us, the vocabulary of an age will always be relational, and the structures in the age are finite.
2. Fraïssé's definition of *age* omits the condition *AP*. He proved results with and without this condition. With *AP*, the limit structures are unique and homogeneous, as in the theorem below. We added *AP* to the definition above because we do not want to consider ages without *AP*, and we do not want to have to say everywhere “age satisfying *AP*”.

Theorem 1.6 (Fraïssé). *Let \mathbf{K} be a countable age. Then there is a countable structure \mathcal{A} , unique up to isomorphism, such that the isomorphism types of finite substructures of \mathcal{A} are exactly the isomorphism types of structures in \mathbf{K} . Moreover, \mathcal{A} is “homogeneous” in the sense that any isomorphism between finite substructures of \mathcal{A} extends to an automorphism of \mathcal{A} .*

For an account of the proof of Theorem 1.6, see the model theory textbook by Hodges [4]. It is not at all difficult. We construct \mathcal{A} as the union of a chain of finite structures \mathcal{A}_s , all isomorphic to elements of \mathbf{K} . We extend, step by step, with the goal of producing a structure that includes copies of all elements of \mathbf{K} as substructures and is homogeneous. The Joint Embedding Property and the Amalgamation Property guarantee that there is always an appropriate next structure.

Definition 3. *For a countable age \mathbf{K} , the structure \mathcal{A} as in Theorem 1.6 is called the Fraïssé limit of \mathbf{K} .*

We want Fraïssé limits that are *computable*. The proof of Theorem 1.6 is effective, given a nice computable list of the structures in the age, and an effective way to determine when one structure in this list embeds in another. We give some definitions to make these things precise. The first definition says what we mean by a nice computable list of structures in \mathbf{K} . In addition to saying how to compute the atomic diagram of each structure, the list gives the full universe, in terms of the standard list of finite sets $(D_n)_{n \in \omega}$.

Definition 4 (Computable representation). *Let τ be a computable relational vocabulary, and let \mathbf{K} be an age consisting of τ -structures. A computable representation of \mathbf{K} is a computable sequence \mathbb{K} such that*

1. *for each i , $\mathbb{K}(i)$ is a pair (e, n) such that φ_e is the characteristic function of the atomic diagram of a structure in \mathbf{K} , and D_n is the universe of this structure,*
2. *for each $C \in \mathbf{K}$, there is some i with first component e such that φ_e is the atomic diagram of a copy of C .*

Note. Informally, we may identify a computable representation \mathbb{K} of \mathbf{K} with the uniformly computable sequence of structures $(C_i)_{i \in \omega}$ such that the first component of $\mathbb{K}(i)$ is a computable index for C_i , but we bear in mind that the second component of $\mathbb{K}(i)$ is an index for the full universe of C_i . Knowing that the first component of $\mathbb{K}(i)$ is e , we can effectively determine whether a given c is in the universe of C_i , but given e , we cannot say that the universe has no more elements beyond those in a certain finite set.

The next definition says when one structure (on the list given by a computable representation \mathbb{K}) can be embedded into another.

Definition 5. Let τ be a computable relational language, and let \mathbf{K} be an age consisting of τ -structures. Suppose that $(C_i)_{i \in \omega}$ is the sequence of structures given by a computable representation \mathbb{K} .

1. The corresponding embedding relation, denoted by $E(\mathbb{K})$, is the set of triples (i, j, f) such that f is an embedding of C_i into C_j .
2. We say that \mathbb{K} has the strong embedding property if $E(\mathbb{K})$ is computable.

Remark. If τ is a finite relational vocabulary, then for any computable representation \mathbb{K} of \mathbf{K} , $E(\mathbb{K})$ is computable. If τ is infinite, this is not always true.

Proposition 1.7. There is a computable representation \mathbb{K} of an age \mathbf{K} (for a computable vocabulary τ) such that $E(\mathbb{K})$ is not even c.e.

Proof sketch. Let τ consist of unary predicates U_n for $n \in \omega$. Let \mathbf{K} be the set of finite τ -structures in which each element satisfies U_n for at most one n . The isomorphism type of a structure in \mathbf{K} is determined by the set of n such that the structure has an element in U_n and the number of elements not in any U_n . We construct a computable representation \mathbb{K} of \mathbf{K} such that $E(\mathbb{K})$ is not c.e. We describe the construction of a uniformly computable sequence $(C_i)_{i \in \omega}$ of τ -structures, with universe specified. The effective construction proceeds in stages. At stage s , we determine, for each of finitely many i , the full universe of C_i and a finite part of the atomic diagram. The isomorphism types of the C_i 's must be exactly those of the structures in \mathbf{K} , and we must satisfy the following requirements.

R_e : W_e is not equal to $E(\mathbb{K})$.

The strategy for R_e is as follows. At stage s , when we first begin work on the requirement, we designate a pair of indices $i, i + 1$, on which we have not yet specified the universe or said anything about the atomic diagrams. We give C_i universe 2 and C_{i+1} universe 3. Let f be the identity function on 2. We vow to put 1 into U_i in both structures, and to put 2 into U_{i+1} in C_{i+1} . We keep 0 out of U_n in C_{i+1} . We vow to keep 0 out of all U_n in C_i unless the triple $(i, i + 1, f)$ appears in W_e . If this happens at stage s , then for the first n such

that we have not already put into the diagram of C_i the statement $\neg U_n(0)$, we add the statement $U_n(0)$.

We continue enumerating the diagrams of structures C_i , making sure that the isomorphism types match those in \mathbf{K} , and satisfying the requirements. So, by definition, \mathbb{K} is a computable representation of \mathbf{K} and $E(\mathbb{K}) \neq W_e$, for all e , which proves the result. \square

Here are the last definitions we shall need in discussing computable Fraïssé limits.

Definition 6. Let \mathbf{K} be an age, with computable representation \mathbb{K} . Let $(C_i)_{i \in \omega}$ be the corresponding sequence of structures. Let \mathcal{A} be a Fraïssé limit of \mathbf{K} .

1. $E(\mathbb{K}, \mathcal{A})$ is the set of pairs (i, f) such that f is an embedding of C_i into \mathcal{A} .
2. \mathcal{A} is effectively homogeneous if the set of finite partial isomorphisms between substructures of \mathcal{A} is computable.

Here is the result that we will use in Section 3.

Theorem 1.8. Let τ be a computable relational language, possibly infinite. Let \mathbf{K} be an age consisting of τ -structures. Suppose that \mathbb{K} is a computable representation of \mathbf{K} with the strong embedding property. Then there is a computable Fraïssé limit \mathcal{A} such that $E(\mathbb{K}, \mathcal{A})$ is computable. In fact, we have a uniform effective procedure for passing from τ , \mathbb{K} and $E(\mathbb{K})$ to $D(\mathcal{A})$ and $E(\mathbb{K}, \mathcal{A})$.

Proof Sketch. The assumptions that \mathbb{K} is a computable representation of \mathbf{K} and that $E(\mathbb{K})$ is computable let us carry out the construction from [4] effectively. Say that $(C_i)_{i \in \omega}$ is the sequence of structures given by \mathbb{K} . We construct the computable Fraïssé limit \mathcal{A} as the union of a uniformly computable sequence of finite structures \mathcal{A}_s , specifying at each step an isomorphism f_s from some C_i onto \mathcal{A}_s . We determine a computable sequence of pairs $(i_s, f_s)_{s \in \omega}$ such that f_s is an isomorphism from C_{i_s} onto \mathcal{A}_s . We know what to put into the diagram of \mathcal{A}_s by looking f_s and the diagram of C_{i_s} .

To see that $E(\mathbb{K}, \mathcal{A})$ is computable, consider f mapping the universe of C_i into \mathcal{A} . For some s , we have $\text{ran}(f) \subseteq \mathcal{A}_s$, and we have specified a function f_s mapping some C_j isomorphically onto \mathcal{A}_s . Let $g = f_s^{-1} \circ f$. Then $(i, f) \in E(\mathbb{K}, \mathcal{A})$ iff $(i, j, g) \in E(\mathbb{K})$.

We have described a uniform procedure that takes the inputs τ , \mathbb{K} and $E(\mathbb{K})$, and effectively produces $D(\mathcal{A})$ and $E(\mathbb{K}, \mathcal{A})$. \square

We defined effective homogeneity. The next result connects it with the relation $E(\mathbb{K}, \mathcal{A})$.

Proposition 1.9. Suppose \mathbf{K} is an age with a computable representation \mathbb{K} and a \mathcal{A} is a computable Fraïssé limit such that $E(\mathbb{K}, \mathcal{A})$ is computable. Then \mathcal{A} is effectively homogeneous.

Proof. We suppose that \mathcal{A} has universe ω . Let f be a finite partial $1 - 1$ function. Find i and g such that $(i, g) \in E(\mathbb{K}, \mathcal{A})$, and let $h = f \circ g$. Now, f is an isomorphism between finite substructures of \mathcal{A} iff $(i, h) \in E(\mathbb{K}, \mathcal{A})$. \square

In [2], Csima et al give necessary and sufficient conditions for an age to give rise to a computable limit structure. They allow function symbols in the vocabulary, and the structures in the age are finitely generated, but not necessarily finite. Even assuming that the vocabulary is relational, the result in [2] does not match Theorem 1.8. The hypotheses of Csima et al are weaker, and the conclusion is also weaker. In particular, the embedding relation is not computable. The result in [2] was inspired by an old result of Goncharov [3] and Peretyat'kin [8], giving necessary and sufficient conditions for a countable homogeneous structure to have a decidable copy. The proof in [2], like those in [3] and [8], involves a priority construction, with guesses at the extension relation, and injury resulting from guesses that are not correct. This precludes effective homogeneity. Theorem 1.8 is much more elementary.

In Section 3, we will construct, by induction, a family of computable limit structures \mathcal{A}_α corresponding to computable ordinals α (really, we will work with notations for ordinals). For each α , we obtain \mathcal{A}_α by applying Theorem 1.8 to a triple of inputs τ_α , \mathbb{K}_α , and $E(\mathbb{K}_\alpha)$. It is straightforward to show that, given the inputs for \mathcal{A}_β for $\beta < \alpha$, we can pass effectively to the inputs for \mathcal{A}_α . We first attempted this construction using the result in [2], where the inputs for \mathcal{A}_α included only a weak substitute for $E(\mathbb{K}_\alpha)$. Passing effectively from the inputs for \mathcal{A}_β for $\beta < \alpha$ to the inputs for \mathcal{A}_α seemed too cumbersome. We were pleased to find that we could apply the more elementary Theorem 1.8.

2 The Hanf number is at most $\beth_{\omega_1^{CK}}$

Our goal in this section is to prove that the Hanf number for Scott sentences of computable structures is at most $\beth_{\omega_1^{CK}}$. The lemma below says that for a computable structure \mathcal{A} , we can replace the Scott sentence, which may not be computable infinitary, by a low level computable infinitary sentence in a larger vocabulary. Let τ be a computable vocabulary, and let \mathcal{A} be a computable τ -structure. From the original proof of the Scott Isomorphism Theorem [11], there is a family of $\mathcal{L}_{\omega_1, \omega}(\tau)$ -formulas $\varphi_{\bar{a}}(\bar{x})$, corresponding to tuples \bar{a} in \mathcal{A} , such that $\varphi_{\bar{a}}(\bar{x})$ defines the orbit of \bar{a} under automorphisms of \mathcal{A} . By Theorem 1.5 (b), we may take $\varphi_{\bar{a}}(\bar{x})$ to be the conjunction of the computable infinitary formulas true of \bar{a} .

Lemma 2.1. *Let τ be a computable vocabulary, and let \mathcal{A} be a computable τ -structure with Scott sentence φ . There is a computable vocabulary $\tau^* \supseteq \tau$ with a c.e. set T of computable infinitary τ^* -sentences (all computable Π_2) such that for any τ -structure \mathcal{B} , $\mathcal{B} \models \varphi$ iff \mathcal{B} has an expansion \mathcal{B}^* satisfying T .*

Proof. The vocabulary τ^* has predicates $P_{\bar{a}}$ for all tuples $\bar{a} \in \mathcal{A}$. We put into T sentences saying the following.

1. $(\forall \bar{x})[P_{\bar{a}}(\bar{x}) \rightarrow \varphi(\bar{x})]$, where $\varphi(\bar{x})$ is a finitary quantifier-free formula true of \bar{a} in \mathcal{A} (this is computable Π_1),
2. $(\forall y) \bigvee_b P_b(y) \ \& \ \bigwedge_b (\exists y) P_b(y)$, where the disjunction and conjunction are over b in \mathcal{A} (this is computable Π_2),
3. $(\forall \bar{x})[P_{\bar{a}}(\bar{x}) \rightarrow ((\forall y) \bigvee_b P_{\bar{a},b}(\bar{x}, y) \ \& \ \bigwedge_b (\exists y) P_{\bar{a},b}(\bar{x}, y))]$, where \bar{a} is a tuple in \mathcal{A} . As for (2), the disjunction and conjunction are over b in \mathcal{A} (this is computable Π_2).

Since \mathcal{A} is computable, it is clear that T is a c.e. set of computable τ^* -sentences, all computable Π_2 or simpler. We show that a τ -structure \mathcal{B} is a model of the Scott sentence φ iff it can be expanded to a model of T .

(\Rightarrow): Suppose \mathcal{B} is a model of the Scott sentence φ . We show that \mathcal{B} can be expanded to a model \mathcal{B}^* of T . For \bar{c} in \mathcal{B} , we put \bar{c} into $P_{\bar{a}}^{\mathcal{B}^*}$ iff \bar{c} satisfies in \mathcal{B} the computable infinitary τ -formulas that were true of \bar{a} in \mathcal{A} . There may be many tuples \bar{a}' in \mathcal{A} satisfying the same computable infinitary τ -formulas as \bar{a} , and \bar{c} will be in all of the corresponding relations $P_{\bar{a}'}^{\mathcal{B}^*}$. We check that \mathcal{B}^* is a model of T . The sentences of type (1) are clearly true. All of the relations P_b are satisfied in \mathcal{B}^* , and each element of \mathcal{B}^* satisfies at least one P_b . Therefore, the sentences of type (2) are true. Supposing that \bar{c} satisfies $P_{\bar{a}}(\bar{u})$ in \mathcal{B}^* , there are elements d satisfying $P_{\bar{a},b}(\bar{c}, x)$, and every element d satisfies one of these $P_{\bar{a},b}(\bar{c}, x)$. Therefore, the sentences of type (3) are true.

(\Leftarrow): Now, suppose that \mathcal{B} has an expansion \mathcal{B}^* satisfying T . We must show that \mathcal{B} satisfies φ . It is convenient to suppose that \mathcal{B}^* is countable. (In case it is not, we take the countable fragment F generated by φ and the sentences of T . We replace \mathcal{B}^* by a countable F -elementary substructure \mathcal{C}^* , and we replace \mathcal{B} by the appropriate reduct \mathcal{C} .) Supposing that \mathcal{B}^* is countable, we show that \mathcal{B} satisfies φ by showing that $\mathcal{A} \cong \mathcal{B}$. Let \mathcal{F} be the set of finite partial functions mapping a non-empty tuple \bar{a} in \mathcal{A} to a tuple \bar{b} in \mathcal{B} such that $\mathcal{B}^* \models P_{\bar{a}}(\bar{b})$. We show that \mathcal{F} has the back-and-forth property. Suppose $f \in \mathcal{F}$ maps \bar{a} to \bar{b} . For any c in \mathcal{A} , there is some d in \mathcal{B} such that $\mathcal{B}^* \models P_{\bar{a},c}(\bar{b}, d)$, so $f \cup \{(c, d)\} \in \mathcal{F}$. For any d in \mathcal{B} , there is some c such that $\mathcal{B}^* \models P_{\bar{a},c}(\bar{b}, d)$. Then $f \cup \{(c, d)\} \in \mathcal{F}$.

We note that the given \mathcal{A} has a computable expansion to a model of T in which, for each \bar{a} , the only tuple in the interpretation of $P_{\bar{a}}$ is \bar{a} itself. There is another expansion of \mathcal{A} to a model of T , in which a tuple \bar{c} is in the interpretation of $P_{\bar{a}}$ just in case \bar{c} satisfies all of the computable infinitary τ -formulas true of \bar{a} . We do not claim that this second expansion is computable, but of course this does not matter. \square

If A is a countable admissible set containing the signature τ and the τ -structure \mathcal{A} , then the set T , formed exactly as above, is c.e. relative to \mathcal{A} , and it consists of very simple sentences in an expanded signature τ^* , where both τ^* and T are in A . Again, \mathcal{B} is a model of the Scott sentence for \mathcal{A} iff it can be expanded to a model of T .

Using Lemma 2.1, we can prove Theorem 1.1.

Proof of Theorem 1.1. From the original Scott sentence φ , in a computable vocabulary τ , we pass to the c.e. set of sentences T in the expanded vocabulary τ^* , where τ^* is still computable. Let φ^* be the conjunction of T . This is a computable infinitary τ^* -sentence. For each $\alpha < \omega_1^{CK}$, the sentence φ has a model \mathcal{B} of cardinality \beth_α . By Lemma 2.1, \mathcal{B} can be expanded to a model \mathcal{B}^* of T , and φ^* . Applying Theorem 1.4 to the computable infinitary τ^* -sentence φ^* , we get the fact that there are arbitrarily large models. By Lemma 2.1, the τ -reducts of these all satisfy φ . \square

In the same way, we see that the Hanf number for Scott sentences of hyperarithmetical structures is at most $\beth_{\omega_1^{CK}}$. In fact, for a countable admissible set A with ordinal γ , the Hanf number for Scott sentences of structures in A is at most \beth_γ .

3 The Hanf number is at least $\beth_{\omega_1^{CK}}$

Recall that an infinite cardinal κ is *characterized* by an $\mathcal{L}_{\omega_1, \omega}$ sentence φ if φ has a model of cardinality κ but does not have a model of cardinality κ^+ . For each $\alpha < \omega_1^{CK}$, we construct a computable structure whose Scott sentence characterizes \beth_α , thus proving that the Hanf number for Scott sentences of computable structures is *exactly* equal to $\beth_{\omega_1^{CK}}$. In fact, we prove the following.

Theorem 3.1. *There exists a partial computable function I such that for each $a \in \mathcal{O}$, $I(a)$ is a tuple of computable indices for several objects, among which are a relational vocabulary τ_a , and the atomic diagram of a τ_a -structure \mathcal{A}_a , with the following features:*

1. the Scott sentence φ_a of the structure \mathcal{A}_a characterizes the cardinal $\beth_{|a|}$, where $|a|$ is the ordinal with notation a ,
2. the vocabulary τ_a contains a unary predicate U_a and a binary relation $<_a$ such that
 - (a) $(U_a, <_a)$ is a dense linear order without endpoints,
 - (b) there is a model \mathcal{B} of φ_a of cardinality $\beth_{|a|}$ such that $(U_a^{\mathcal{B}}, <_a^{\mathcal{B}})$ has a co-final sequence of order type $\beth_{|a|}$.

We define I by computable transfinite recursion on ordinal notation. For each a , $I(a)$ is a tuple of computable indices for the following:

1. the vocabulary τ_a ,
2. a representation \mathbb{K}_a of an age \mathbf{K}_a ,
3. $E(\mathbb{K}_a)$,

4. the atomic diagram of \mathcal{A}_a , the Fraisse limit of \mathbf{K}_a ,
5. $E(\mathbb{K}, \mathcal{A}_a)$.

The structure \mathcal{A}_a along with the relation $E(\mathbb{K}, \mathcal{A}_a)$ are obtained by applying the uniform effective procedure of Theorem 1.8 to τ_a , \mathbb{K}_a and $E(\mathbb{K}_a)$. We must arrange that the Scott sentence φ_a for \mathcal{A}_a characterizes the cardinal $\beth_{|a|}$.

Base case. Recall that 1 is the unique notation for 0. We describe $I(1)$. The vocabulary τ_1 consists of unary relation symbols U_1 and Q_q for $q \in \mathbb{Q}$, plus the binary relation symbol $<_1$. We want \mathcal{A}_1 to be an expansion of $(\mathbb{Q}, <)$ in which the interpretation of U_1 consists of all rationals, and the interpretation of Q_q consists just of q . The Scott sentence of \mathcal{A}_1 has no uncountable model. The age \mathbf{K}_1 consists of finite substructures of \mathcal{A}_1 , including the empty structure. It is not difficult to see that there is a computable representation \mathbb{K}_1 of \mathbf{K}_1 for which the embedding relation $E_1 = E(\mathbb{K}_1)$ is computable. We apply the uniform effective procedure from Theorem 1.8 to get a computable limit structure \mathcal{A}_1 such that $E(\mathbb{K}_1, \mathcal{A}_1)$ is also computable.

Inductive step. We define $I(a)$, assuming that we have previously defined $I(b)$ for all $b <_{\mathcal{O}} a$, and $a \neq 1$. Recall that for $a \in \mathcal{O}$, $|a|$ is the ordinal with notation a . We split the construction into two cases, depending on whether $|a|$ is a successor ordinal or a limit ordinal.

3.1 Successor Ordinals

In this subsection, we suppose that I has been defined on all $b \leq_{\mathcal{O}} a$ so that the conditions of Theorem 3.1 are satisfied. We suppose that $I(a)$ is a code for a quintuple of indices for τ_a , \mathbb{K}_a , $E(\mathbb{K}_a)$, $D(\mathcal{A}_a)$, and $E(\mathbb{K}_a, \mathcal{A}_a)$. The structure \mathcal{A}_a is the Fraïssé limit, which is obtained from \mathbb{K}_a and $E(\mathbb{K}_a)$ as in Theorem 1.8, and the Scott sentence φ_a of \mathcal{A} characterizes the cardinal $\beth_{|a|}$.

By the induction hypothesis, we have a unary predicate U_a and a binary relation $<_a$ such that

- (a) $(U_a, <_a)$ is a dense linear order without endpoints (in any model of φ_a),
- (b) there is a model \mathcal{B} of φ_a of size $\beth_{|a|}$ such that $(U_a^{\mathcal{B}}, <_a^{\mathcal{B}})$ contains a co-final sequence of order type $\beth_{|a|}$.

Then we inductively extend the definition of I to $b = 2^a$, where $|b| = |a| + 1$. The construction is a modified version of that in [12]. We let τ_b be the vocabulary $\tau_a \cup \{V, M, U_b, P, F, <_b\}$, where V , M , and U_b are unary predicates, $<_b$ is a binary predicate and F is a ternary predicate. We suppose that the symbols V , M , U_b , P , F , and $<_b$ are new, not in τ_a . We first describe \mathbf{K}_b and show that it is an age. Then we consider the computable indices that make up $I(b)$.

We let \mathbf{K}_b be the collection of all finite τ_b -structures that satisfy the conjunction of the following:

1. The domain is the disjoint union of V , M , U_b . Think of V as a set of vertices and M as a set of edge-colors and U_b as a set of vertex-colors.
2. $M \upharpoonright \tau_a$ is a structure in \mathbf{K}_a . In particular, there is a linear order $<_a$ defined on a subset U_a of M .
3. All relations in τ_a are void outside of M .
4. The predicate P defines a vertex-coloring on V with values in U_b . That is, for each $v \in V$, there is *at most* one $u \in U_b$ such that $P(v, u)$.
5. The predicate F defines an edge-coloring on $[V]^2 \setminus \{(v, v) | v \in V\}$. This time, the colors are elements of U_a ; i.e., for each pair $v_0, v_1 \in V$, there is *at most* one $u \in U_a$ such that $F(v_0, v_1, u)$ and $F(v_1, v_0, u)$. We will just write $F(v_0, v_1) = u = F(v_1, v_0)$.
6. $<_b$ is a linear order on U_b .

The next property is the one that drives the construction.

7. For any triple of distinct elements $v_0, v_1, v_2 \in V$, if $F(v_0, v_1) \neq F(v_0, v_2)$, then

$$F(v_1, v_2) = \min\{F(v_0, v_1), F(v_0, v_2)\}, \quad (\star)$$

where \min is according to the $<_a$ -ordering.

Otherwise, $F(v_1, v_2) >_a F(v_0, v_1) = F(v_0, v_2)$.

Remark. The collection \mathbf{K}_b described above differs from the collection $K(\mathcal{M})$ in [12] in the following respects:

1. The set U_b and the projection P are missing in $K(\mathcal{M})$. The reason it is introduced here is that we need it to carry out the induction.
2. Here the set M is finite and its restriction to τ_a is a (finite) structure in \mathbf{K}_a . In [12], the set \mathcal{M} is infinite, and its restriction on some vocabulary τ satisfies an $\mathcal{L}_{\omega_1, \omega}(\tau)$ -sentence φ .
3. The requirement that P and F are total functions defined on their corresponding domains has been relaxed to solely requiring that they take at most one value. The reason is that we need \mathbf{K} to satisfy *HP*. This is not the case in [12]. Nevertheless, in the Fraïssé limit, both P and F will be total functions, not just partial functions.
4. The empty structure belongs to \mathbf{K}_b , since it also belongs to \mathbf{K}_a .

Lemma 3.2. \mathbf{K}_b satisfies *HP*, *JEP* and *AP*.

Proof. The hereditary property follows immediately from the definition of \mathbf{K}_b . We will sketch the proof just for AP . We get JEP for free, since $\emptyset \in \mathbf{K}_b$. Let $A, B, C \in \mathbf{K}_b$ where C is a substructure of A and B . We need an amalgam $D \in \mathbf{K}_b$ with embeddings $f : A \rightarrow D$ and $g : B \rightarrow D$ such that f and g agree on C . Since \mathbf{K}_a satisfies AP , we can take the reducts to τ_a of M^A and M^B over M^C . Let D_1 be the τ_b -structure with M^{D_1} equal to the resulting amalgam, and with V^{D_1} and $U_b^{D_1}$ empty. We may suppose that D_1 extends M^C , and that it is disjoint from $V^A \cup U_b^A$ and $V^B \cup U_b^B$. Let f_1 embed M^A into M^D , and let g_1 embed M^B into M^D , where f_1 and g_1 agree with the identity function on M^C .

Next, using the argument from Lemma 4.9 of [12], we amalgamate $V^A \cup M^{D_1}$ and $V^B \cup M^{D_1}$ over $V^C \cup M^{D_1}$, considering these as τ_b -structures, with the appropriate interpretations of F . Let D_2 be the amalgam structure, with $U_b^{D_2}$ empty. We may suppose that D_2 extends D_1 and that it is disjoint from U_b^A and U_b^B . Let f_2 embed $D_1 \cup V^A$ into D_2 and let g_2 embed $D_1 \cup V^B$ into D_2 , where f_2 and g_2 agree with the identity function on $D_1 \cup V^C$. The argument from Lemma 4.9 of [12] shows that $V^{D_2} = V^A \cup V^B$. In the amalgam D_2 , although no new points are added in forming V^{D_2} , it is possible that some new points are added in forming M^{D_2} (that is, M^{D_2} may have elements not in M^{D_1}).

Finally, we amalgamate $D_2 \cup U_b^A$ and $D_2 \cup U_b^B$ over $D_2 \cup U_b^C$, considering these as τ_b structures, with the appropriate interpretations of P . The amalgam structure is the desired D . We may suppose that D extends $D_2 \cup U_b^C$. We let f_3 embed $D_2 \cup U_b^A$ into D , and we let g_3 embed $D_2 \cup U_b^B$ into D , where f_3 and g_3 agree with the identity on $D_2 \cup U_b^C$. Then D is an extension of C . Let f be the restriction of f_3 to A , and let g be the restriction of g_3 to B . Then f and g are embeddings that agree on C . \square

Assuming that the symbols in τ_a are each marked by some notation $a' \leq_O a$, we mark the finitely many new symbols by b . We pass from an index for τ_a to an index for τ_b . We can easily pass from a computable representation \mathbb{K}_a of \mathbf{K}_a to a computable representation \mathbb{K}_b of \mathbf{K}_b . Say that $(C_i)_{i \in \omega}$ is the sequence of structures given by \mathbb{K}_a . For \mathbb{K}_b , we will have sequence of structures $(D_i)_{i \in \omega}$ such that for each $i = \langle i_1, i_2 \rangle$, $M^{D_i} = C_{i_1}$. The other parts of D_i , namely, V^{D_i} and U^{D_i} , are finite, with the relations F , P , and $<_b$ to be determined. There are only finitely many symbols whose interpretation in D_i is not determined by C_{i_1} .

We can compute $E(\mathbb{K}_b)$, using $E(\mathbb{K}_a)$ and \mathbb{K}_b . To determine whether a finite function f is an embedding of D_i into D_j , we first check whether the appropriate restriction of f embeds C_{i_1} into C_{j_1} , and we then check that the finitely many further relations, involving new elements, are preserved. From \mathbb{K}_b and $E(\mathbb{K}_b)$, we compute $D(\mathcal{A}_b)$ and $E(\mathbb{K}_b, \mathcal{A}_b)$, as in Theorem 1.8. Thus, we have $I(b)$, with computable indices for τ_b , \mathbb{K}_b , $E(\mathbb{K}_b)$, $D(\mathcal{A}_b)$, and $E(\mathbb{K}_b, \mathcal{A}_b)$.

Theorem 3.3. *There is a computable Fraïssé limit \mathcal{A}_b of \mathbf{K}_b , with Scott sentence φ_b , such that*

1. $M^{\mathcal{A}_b} \upharpoonright \tau_a$ is isomorphic to the τ_a -structure \mathcal{A}_a ,

2. φ_b characterizes the cardinal $\beth_{|b|}$,
3. $(U_b, <_b)$ is a dense linear order without endpoints, and
4. φ_b has a model \mathcal{B} of size $\beth_{|b|}$ such that $(U_b^{\mathcal{B}}, <_b^{\mathcal{B}})$ contains a co-final sequence of order type $\beth_{|b|}$.

Proof. Clause (1) follows from the inductive hypothesis. Clause (2) follows from the proofs of Theorems 4.13 and 4.14 in [12]¹, modified to include U_b and P . As in the proof of Lemma 3.2, U_b^D and P^D are defined in the amalgam to be the union of $U_b^B \cup U_b^C$ and $P^B \cup P^C$, respectively. The proof goes through because of disjoint amalgamation.

Clause (3) follows from the usual proof that the Fraïssé limit of finite linear orders yields a dense linear order without endpoints. Clause (4) follows from the fact that we can organize a sequence of $\beth_{|b|}$ -many amalgamation triples $(A_i, B_i, C_i)_{i < \beth_{|b|}}$, where the structures A_i , B_i , and C_i are linearly ordered, A_i and B_i are finite, C_i may be infinite, and C_{i+1} is the amalgam of B_i, C_i over A_i , so that $C_* = \bigcup_{i < \beth_{|b|}} C_i$ contains a co-final sequence of order type $\beth_{|b|}$. \square

3.2 Limit Ordinals

Assume $|a|$ is a limit ordinal. Then a has the form $3 \cdot 5^e$, where φ_e is a total recursive function with values $\varphi_e(n) = a_n$ such that $a_n <_{\mathcal{O}} a_{n+1}$ for all n , and $|a| = \lim_n |a_n|$. Without loss of generality, we may suppose that $|a_n|$ is a successor ordinal.

Let τ_a be the union of the τ_{a_n} 's, together with the new unary predicates U_a and Q_n , for $n \in \omega$, and the new binary predicate $<_a$. Let \mathbf{K}_a consist of all τ_a -structures (with universe a finite subset of ω) such that

1. $Q_n \upharpoonright \tau_{a_n}$ is a structure in \mathbf{K}_{a_n} ,
2. the relations in τ_{a_n} are void outside Q_n ,
3. $U_a = \bigcup_n U_{a_n}$, where each U_{a_n} is the subset of Q_n linearly ordered by $<_{a_n}$, and
4. for $u_0, u_1 \in U_a$, we define $u_0 <_a u_1$ iff either there exists n such that $Q_n(u_0)$ and $Q_n(u_1)$ and $u_0 <_{a_n} u_1$, or there exist n, m with $n < m$ such that $Q_n(u_0)$ and $Q_m(u_1)$.

Clearly, \mathbf{K}_a satisfies *HP*, *JEP*, and *AP*. Given indices for \mathbb{K}_{a_n} and $E(\mathbb{K}_{a_n})$, for all n , we can produce a computable representation \mathbb{K}_a of \mathbf{K}_a such that $E(\mathbb{K}_a)$ is also computable. We partition ω into disjoint sets Q_n . For each n , let p_n be the function mapping the elements of ω 1–1 onto the elements of Q_n , in order. We have a computable list of all finite partial functions $(\sigma_i)_{i \in \omega}$ from ω to ω . Let $(C_i^{a_n})_{i \in \omega}$ be the sequence of structures given by \mathbb{K}_{a_n} , and let $(C_j^a)_{j \in \omega}$ be the sequence of structures given by \mathbb{K}_a . We use σ_i to determine C_i^a . For each

¹See also Remark 4.28 and Theorem 4.29 in [12].

$n \in \text{dom}(\sigma_i)$, we put into $Q_n^{C_i}$ a copy of the structure $C_j^{a_n}$, where $j = \sigma_i(n)$. As our isomorphism taking $C_j^{a_n}$ to the copy, we take the restriction of p_n to the universe of $C_j^{a_n}$. We complete the structure C_i^a in the only way possible, letting $U_a^{C_i^a}$ be the union of the sets $U_{a_n}^{C_j^{a_n}}$, for $j = \sigma_i(n)$, and defining the ordering $<_a$ as prescribed.

It is clear that \mathbb{K}_a is a computable representation of \mathbf{K}_a . Moreover, $E(\mathbb{K}_a)$ is computable. We have $(i, j, f) \in E(\mathbb{K}_a)$ iff $\text{dom}(\sigma_i) \subseteq \text{dom}(\sigma_j)$, and for each $n \in \text{dom}(\sigma_i)$, if $\sigma_i(n) = i'$ and $\sigma_j(n) = j'$, and f' is the finite partial function such that $f \circ p_n = p_n \circ f'$, then $(i', j', f') \in E(\mathbb{K}_{a_n})$. From \mathbb{K}_a and $E(\mathbb{K}_a)$, we obtain the Fraïssé limit $D(\mathcal{A}_a)$ and $E(\mathbb{K}_a, \mathcal{A}_a)$.

1. $Q_n^{\mathcal{A}_a} \upharpoonright \tau_{a_n} \cong \mathcal{A}_{a_n}$,
2. φ_a characterizes the cardinal $\beth_{|a|}$,
3. $(U_a, <_a)$ is a dense linear order without endpoints, and
4. there is a model \mathcal{B} of size $\beth_{|a|}$ such that $(U_a^{\mathcal{B}}, <_a^{\mathcal{B}})$ contains a co-final sequence of order type $\beth_{|a|}$.

The last statement is true because, by the induction hypothesis, for each n , there exists a model \mathcal{B}_{α_n} such that $(U_{a_n}^{\mathcal{B}_{\alpha_n}}, <_{a_n}^{\mathcal{B}_{\alpha_n}})$ contains a co-final sequence of order type $\beth_{|a_n|}$. Let \mathcal{B} be the τ_a -structure that agrees with each \mathcal{B}_{α_n} on Q_n . Then $<_a^{\mathcal{B}}$ contains a co-final sequence of order type $\beth_{|a|}$. It also follows from what we have said above that τ_a is a computable relational vocabulary and \mathcal{A}_a is a computable τ_a -structure with Scott sentence φ characterizing $\beth_{|a|}$. To complete the proof of Theorem 3.1, we observe the following.

Corollary 3.4. *We have a partial computable function I that, for each $a \in \mathcal{O}$, gives indices for τ_a , \mathbb{K}_a , $E(\mathbb{K}_a)$, $D(\mathcal{A}_a)$, and $E(\mathbb{K}_a, \mathcal{A}_a)$, where \mathcal{A}_a has a Scott sentence φ_a characterizing $\beth_{|a|}$.*

Corollary 3.5. *The Hanf number for Scott sentences of hyperarithmetical structures is $\beth_{\omega_1^{CK}}$.*

We already remarked that $\beth_{\omega_1^{CK}}$ is an upper bound on the Hanf number. The computable structures \mathcal{A}_a from the previous section witness that it is also a lower bound.

Remark. If A is a countable admissible set with ordinal γ , then the Hanf number for Scott sentences for structures in A is \beth_γ . We already remarked that \beth_γ is an upper bound. Using essentially the same construction as in this section, we could determine a function I , Σ -definable in A , taking each ordinal $\alpha < \gamma$ to a tuple of elements of A , consisting of a vocabulary τ_α , a representation \mathbb{K}_α of an age \mathbf{K}_α , the embedding relation $E(\mathbb{K}_\alpha)$, the limit structure \mathcal{A}_α obtained effectively from \mathbb{K}_α and $E(\mathbb{K}_\alpha)$, and the relation $E(\mathbb{K}_\alpha, \mathcal{A}_\alpha)$.

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